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COMPLETENESS OF SOME NORMAL MODAL LOGICS

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Abstract. In this work, we provide simple proofs for the completeness of the normal modal logics **B**, **K**5 and **G**.

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1. Introduction

The completeness of the modal logic $S4$ for all topological spaces, in which the modal operator box, $\Box$, is interpreted as interior, was proved by McKinsey and Tarski in [9]. In this seminal work, they showed that the Stone representation theorem for Boolean algebras [13] extends to algebra with operators to give a topological semantics for propositional modal logic in which the necessity operation box, $\Box$, is modeled by taking the interior of an arbitrary subset of a topological space. In this theorem, the embedding function assigns each elements of Boolean algebra to an ultrafilter of that algebra. This topological interpretation is extended to arbitrary theories of first-order logic by way of different approaches; for example, see [1], [2], [7] and [8]. There are several methods to prove completeness of modal logics; these are the classical Kripke semantics, the Fitting tableaux semantics (see [5] and [6]) and the topological semantics (see [11]). All these semantics are analyzed in great details in [10], and partially in [12].

2. Preliminaries

In this section, we introduce basic notations of propositional modal logic; for more, for example, see [4] and [3].

The basic modal language can be defined different ways, depending on the choice of Boolean connectives and modal operators box , $\Box$, and diamond, $\Diamond$. Throughout $\mathfrak{B}$ will denote the set of propositional letters or atomic propositions of propositional calculus, and the constants "T" and "$\perp$" will mean "true" and "false", respectively.

Definition 1. The basic modal language $\mathcal{L}_{\Box}$ consists of (infinitely countable set) $\mathfrak{B}$ and the connectives $\perp, \rightarrow, \Box$.

Definition 2. Formulas of $\mathcal{L}_\Box$ are defined inductively as follows:

(a) Every $p \in \mathfrak{B}$ is a formula.
(b) The constant $\perp$ is a formula.
(c) If φ and ψ are formulas, then so is $\varphi \to \psi$.
(d) If φ is a formula, then so is $\Box\varphi$.

We denote by $\mathfrak{I}_\Box$ the set of formulas of $\mathcal{L}_\Box$.

The following standard abbreviations are to be noted:

$$\neg\varphi = \varphi \to \perp,\ \ \mathrm{T} = \neg\perp,$$
$$\varphi \wedge \psi = \neg(\varphi \to \neg\psi),$$
$$\varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi), \text{ and } \Diamond\varphi = \neg\Box\neg\varphi.$$

Definition 3. A subset of $\mathcal{L}$ of $\mathfrak{I}_\Box$ is a logic, if it includes all classical tautologies and closed under Modus Ponens, i.e., for all φ, $\psi \in \mathfrak{I}_\Box$, $\varphi \in \mathcal{L}$ and $\varphi \to \psi \in \mathcal{L}$ implies $\psi \in \mathcal{L}$.

Definition 4. A logic $\mathcal{L}$ is normal if it contains the schema

$$(K)\ \ \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$$

and is closed under the Necessity Rule

$$(N)\ \ \textit{for each } \varphi \in \mathfrak{I}_\Box,\ \varphi \in \mathcal{L} \textit{ implies } \Box\varphi \in \mathcal{L}.$$

The smallest (minimal) normal logic **K** contains all tautologies, (K), (N), and *Dual* : $\Diamond\varphi = \neg\Box\neg\varphi$ and is closed under modus ponens and uniform substitution.

Various logics are produced by adding to **K** suitable constraints. Constraints that interest us are the following:

$$B:\ \varphi \to \Box\Diamond\varphi,\ \ E:\ \Diamond\varphi \to \Box\Diamond\varphi,\ \ G:\ \Diamond\Box\varphi \to \Box\Diamond\varphi.$$

In Section 3, we shall prove completeness of the logics $B = \mathbf{K} + B$, $\mathbf{K}5 = \mathbf{K} + \mathbf{E}$, and $\mathbf{G} = \mathbf{K} + G$.

Definition 5. A relational structure (also a Kripke model or simply a modal model) is a triple $\mathfrak{M} = \langle W, R, V\rangle$ where W is a nonempty set (of possible worlds), R is a binary relation on W (called the accessibility relation) and V is a function from $\mathfrak{B}$ to $P(\mathfrak{B})$ (called a valuation mapping). The pair $\langle W, R\rangle$ is called a Kripke frame or a frame and is denoted by $\mathfrak{I}$.

Definition 6. Truth of a modal formula φ at W in a model $\mathfrak{M} = \langle W, R, V\rangle$, denoted by $\mathfrak{M}, w \models \varphi$, is defined inductively as follows:

- $\mathfrak{M}, w \models p$ iff $w \in V(p)$ for $p \in P$.
- $\mathfrak{M}, w \models \mathrm{T}$ and $\mathfrak{M}, w \not\models \perp$.
- $\mathfrak{M}, w \models \neg\varphi$ iff $\mathfrak{M}, w \not\models \varphi$.
- $\mathfrak{M}, w \models \varphi \wedge \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$.
- $\mathfrak{M}, w \models \varphi \vee \psi$ iff $\mathfrak{M}, w \models \varphi$ or $\mathfrak{M}, w \models \psi$.
- $\mathfrak{M}, w \models (\varphi \to \psi)$ iff $(\mathfrak{M}, w \models \varphi \Rightarrow \mathfrak{M}, w \models \psi)$.
- $\mathfrak{M}, w \models \Box\varphi$ iff $\forall u \in W(wRu \Rightarrow \mathfrak{M}, w \models \varphi)$.
- $\mathfrak{M}, w \models \Diamond\varphi$ iff $\exists u \in W(wRu \wedge \mathfrak{M}, w \models \varphi)$.

If a formula φ is true at all points of a model $\mathfrak{M}$, then it is said to be true in $\mathfrak{M}$ and is denoted by $\mathfrak{M} \models \varphi$. Otherwise, it is said to be false and denoted by $\mathfrak{M} \not\models \varphi$.

Definition 7. A formula φ is said to be valid in a frame $\mathfrak{I} = < W, R >$, or $\mathfrak{I}$ validates φ, denoted by $\mathfrak{I} \models \varphi$, if for all valuations $V : \mathfrak{B} \to P(\mathfrak{B})$, $< W, R, V > \models \varphi$.

We shall need some properties of the accessibility relation R in the sequel.

Definition 8. Let $\mathfrak{I} = < W, R >$ be a frame. Then $\mathfrak{I}$ is

(1) serial if R is serial:

$$\forall w \in W\, \exists u \in W : wRu.$$

(2) Euclidean if R is Euclidean:

$$\forall w, u, s \in W (wRu \wedge wRs \Rightarrow uRs).$$

(3) weakly directed if R is weakly directed:

$$\forall w, u, s \in W ((wRu \wedge wRs) \Rightarrow \exists \in W (uRt \wedge sRt)).$$

Definition 9. Let $\mathfrak{M} = < W, R, V >$ be a model and let U be a subset of W. Then U is definable in $\mathfrak{M}$ if $U = (\varphi)^{\mathfrak{M}} = \{w \in W \mid \mathfrak{M}, w \models \varphi\}$, for some $\varphi \in \mathfrak{I}$. Let $\mathfrak{C}$ be a class of frames. A subset S of $\mathfrak{I}$ defines $\mathfrak{C}$ if

$$\mathfrak{I} \in \mathfrak{C} \Leftrightarrow \forall \varphi \in S(\mathfrak{I} \models \varphi).$$

We shall see that logics **B**, **K**5, and **G** define the class of symmetric frames, the class of Euclidean frames, and the class of weakly directed frames, respectively.

Definition 10. Let $\mathcal{L}$ be a logic and $\mathfrak{C}$ a class of frames. $\mathcal{L}$ is said to be sound with respect to $\mathfrak{C}$ if, for every $\varphi \in \mathfrak{I}$, $\vdash_{\mathcal{L}} \Rightarrow \mathfrak{C} \models \varphi$ and $\mathcal{L}$ is said to be complete with respect to $\mathfrak{C}$ if, for each $\varphi \in \mathfrak{I}$, $\mathfrak{C} \models \varphi \Rightarrow \vdash_{\mathcal{L}} \varphi$.

For example, it is shown that S4 is sound and complete with respect to the class of all reflexive and transitive frames. It is this work that motivated researches in this direction.

3. Soundness and Completeness of B, K5 and G

Showing a logic is complete with respect to a class $\mathfrak{C}$ can be done by so-called "canonical model method". What we have to do is to showing that the canonical model of the logic is in the class $\mathfrak{C}$. Showing soundness is in general straightforward, and we shall carry out it only for **B**. Before, we copy out some definitions and results.

Definition 11. Let $\mathcal{L}$ be a logic. Then $\vdash_{\mathcal{L}} \varphi$ means that φ is derivable in $\mathcal{L}$. If Σ is a set of $\mathcal{L}$-formulas, we write $\Sigma \vdash \varphi$ if $\vdash (\psi_1 \wedge \cdots \wedge \psi_k) \to \varphi$ for some finite formulas $\psi_1, \cdots, \psi_k$ from Σ. And if $\Im$ is a frame, we write $\Im \models \Sigma$ for $\Im \models \varphi$ for every $\varphi \in \Sigma$.

Definition 12. A set of formulas Σ is consistent provided $\Sigma \nvdash \bot$. Σ is a maximally consistent set if Σ is consistent and for each $\varphi \in \Im$, either $\varphi \in \Sigma$ or $\neg\varphi \in \Sigma$. Equivalently, Σ is maximally consistent if maximally consistent Σ is consistent and every Γ such that $\Sigma \subseteq \Gamma$ is inconsistent.

Proposition 1. Let Σ be an maximally consistent set. Then the following hold:

(1) If $\vdash \varphi$, then $\varphi \in \Sigma$.
(2) $\neg\varphi \in \Sigma$ iff $\varphi \notin \Sigma$.
(3) If $\varphi \in \Sigma$ and $\varphi \to \psi \in \Sigma$, then $\psi \in \Sigma$.
(4) $\varphi \wedge \psi \in \Sigma$ iff $\varphi \in \Sigma$ and $\psi \in \Sigma$.
(5) $\varphi \vee \psi \in \Sigma$ iff $\varphi \in \Sigma$ or $\psi \in \Sigma$.

Lemma 1. (***Lindenbaum's Lemma***) Every consistent set can be extended to a maximally consistent set.

Definition 13. The canonical model for the minimal modal logic **K** is the model $\mathfrak{M} = < W^C, R^C, V^C >$ where

(1) $W^C = \{\Sigma \mid \Sigma$ is a maximally consistent set$\}$.
(2) $\Sigma R^C \Delta$ iff $\Sigma^{\Box} = \{\varphi \mid \Box\varphi \in \Sigma\} \subseteq \Delta$.
(3) $V^C = \{\Sigma \mid p \in \Sigma\}$.

Remark 1 The set of worlds W^C in the canonical model of **K** is a superset of the set of possible worlds in the canonical model of any other logical system. Thus, if we vary the system, we modify the class of maximally consistent sets, hence we get different canonical models. So proving completeness of a logic is basically showing that if we take a subclass of **K**-worlds, we shall see that the accessibility relation R^C defined in Definition 3.4(2) will turn out to be serial, transitive, reflexive, Euclidean etc. with respect to that subclass, depending on logic.

Theorem 1. The logic $\mathbf{B} = \mathbf{K} + \mathrm{E}$ is sound and complete with respect to the class of all symmetric models, i.e., models whose frame are symmetric.

Proof. Soundness: Let $\mathfrak{M} = < W, R, V >$ be any symmetric model, and let $w \in W$. Suppose that $\mathfrak{M}, w \models \varphi$ such that wRu. Since R is symmetric, we have uRw. As φ is true at W, $\Diamond\varphi$ is true at u for any u such that wRu. But this means that $\Box\Diamond\varphi$ is true at w. Thus, $\varphi \to \Box\Diamond\varphi$ is valid on all symmetric frames. Now according to the well-known result:

If Σ is any set of modal formulas and $< W, R >$ is a frame on which each formula in Σ is valid, then every theorem of $\mathbf{K} + \Sigma$ is valid on $< W, R >$. If follows that **B** is sound with respect to the class of all symmetric frames.

Completeness: To show the completeness of **B** with respect to the class of all symmetric models, it will be enough to show that the canonical model for **B** is a symmetric model. We do this by showing that the relation R is symmetric when consistency is to be **B**-consistency. R is symmetric if and only if xRy implies yRx for any x, y. Thus, the accessibility relation R^C in the canonical model of **B** is symmetric if and only if the following holds:

If $\Sigma R^C \Delta$ then $\Delta R^C \Sigma$ for any maximally consistent set Σ, Δ. But this is translated into the following by Definition 3.4(2):

If $\Sigma^{\Box} = \{\varphi \mid \Box\varphi \in \Sigma\} \subseteq \Delta$, then $\Delta^{\Box} = \{\varphi \mid \Box\varphi \in \Delta\} \subseteq \Sigma$. But we can give another reformulation of $\{\varphi \mid \Box\varphi \in \Delta\} \subseteq \Sigma$ as follows:

If $\varphi \notin \Sigma$ then $\Box\varphi \notin \Delta$.

Thus R^C will be symmetric if $\{\varphi \mid \Box\varphi \in \Sigma\} \subseteq \Delta$, then $\varphi \notin \Sigma \Rightarrow \Box\varphi \notin \Delta$. Now suppose $\{\varphi \mid \Box\varphi \in \Sigma\} \subseteq \Delta$ and $\varphi \notin \Sigma$. Then by maximality of Σ (see Proposition 3.1(2)), we have $\neg\varphi \in \Sigma$. From **B**: $\varphi \to \Box\Diamond\varphi$, we get $\neg\varphi \to \Box\Diamond\neg\varphi$ (**B** is closed under uniform substitution). Now from Proposition 3.1(3) we have that $\neg\varphi \to \Box\Diamond\neg\varphi \in \Sigma$ and $\neg\varphi \in \Sigma$ implies $\Box\Diamond\neg\varphi \in \Sigma$. But $\{\varphi \mid \Box\varphi \in \Sigma\} \subseteq \Delta$ means $\Diamond\neg\varphi \in \Delta$, and by Dual: $\Diamond = \neg\Box\neg$, this means that $\neg\Box\varphi \in \Delta$. Hence by maximality of Δ we get $\Box\varphi \notin \Delta$, and the proof is complete.

Remark 2 Using Dual $\Diamond = \neg\Box\neg$, we can rewrite $E : \Diamond\varphi \to \Box\Diamond\varphi$ as $\neg\Box\neg\varphi \to \Box\neg\Box\neg\varphi$, and substituting $\neg\varphi$ for φ and applying the double law for negation, we obtain the equivalent of $E : \neg\Box\varphi \to \Box\neg\Box\varphi$.

Lemma 2. Let $\mathcal{L}$ be a normal modal logic. If $\neg\Box\varphi \to \Box\neg\Box\varphi$, then the canonical model of $\mathcal{L}$ is Euclidean. In other words, $\mathcal{L}$ is complete with respect to the class of all Euclidean models. f all symmetric models, i.e., models whose frame are symmetric.

Proof. Suppose that $\vdash_{\mathcal{L}} \neg\Box\varphi \to \Box\neg\Box\varphi$. We want to show that for maximally consistent sets Σ, Δ and Γ, if $\Sigma R^C \Delta$ and $\Sigma R^C \Gamma$ then $\Delta R^C \Gamma$.

Suppose that $\Sigma R^C \Delta$ and $\Sigma R^C \Gamma$. Then by definition of R^C, we have $\{\varphi \mid \Box\varphi \in \Sigma\} \subseteq \Delta$ and $\{\varphi \mid \Box\varphi \in \Delta\} \subseteq \Gamma$. Now suppose that $\Box\varphi \in \Delta$. If $\varphi \notin \Gamma$, then $\neg\varphi \in \Gamma$ by maximality of Γ. This implies that $\Box\varphi \notin \Sigma$, and hence $\neg\Box\varphi \in \Sigma$ again by maximality of Σ. Since $\neg\Box\varphi \to \Box\neg\Box\varphi \in \Sigma$ by hypothesis, and Σ is closed under Modus Ponens (see Proposition 3.1(3)), it follows that $\Box\neg\Box\varphi \in \Sigma$. This implies that $\neg\Box\varphi \in \Delta$, a contradiction with the fact that $\Box\varphi \in \Delta$. Hence $\varphi \in \Gamma$, as desired.

Now let us put $\mathcal{L} = \mathbf{K}5 = \mathbf{K} + E$. Then we have the following theorem whose proof follows from Lemma 3.2.

Theorem 2. The logic $\mathbf{K}5 = \mathbf{K} + \mathrm{E}$ is sound and complete with respect to the class of all Euclidean models.

Finally, we prove that the logic $\mathbf{G} = \mathbf{K} + G$ where $G : \Diamond\Box\varphi \to \Box\Diamond\varphi$ is sound and complete. The axiom schema G is translated by the accessibility relation which is weakly directed. Such a relation is also called incestual in [4] and convergent in [12]. These relations are of interest in modeling belief and knowledge.

Theorem 3. The logic $\mathbf{G} = \mathbf{K} + \mathrm{G}$ is sound and complete with respect to the class of all weakly directed models.

Proof. We shall prove that the canonical model for **G** is an incestual model. Recall that a relation R is incestual if and only if $\forall w, u, s \in W((wRu \wedge wRs) \Rightarrow \exists t \in W(uRt \wedge sRt))$. In terms of R^C, this is translated as:

(1) $\{\varphi \mid \Box\varphi \in \Sigma\} \subseteq \Delta$,

(2) $\{\varphi \mid \Box\varphi \in \Sigma\} \subseteq \Gamma$. Then

(3) $\{\varphi \mid \Box\varphi \in \Delta\} \cup \{\varphi \mid \Box\varphi \in \Gamma\}$

is consistent. Here (1) means $\Sigma R^C \Delta$, (2) means $\Sigma R^C \Gamma$, and (3) implies that the set of possible worlds accessible from Δ and the set of worlds accessible from Γ must have at least a member in common. Now we are proving that R^C is incestual.

Suppose to the contrary that the set (3) is not consistent. Then for some $\Box\varphi \in \Delta$, some $\Box\psi \in \Gamma$, the set $\{\varphi, \psi\}$ is not consistent, i.e.,

(a) $\vdash_G \neg(\varphi \wedge \psi)$.

We have the following steps:

1. $\Diamond\Box\varphi \in \Sigma$	(by (1))
2. $\Diamond\Box\psi \in \Sigma$	(by $\Box\varphi \in \Gamma$ and $\Sigma R^C\Gamma$)
3. $\Box\Diamond\psi \in \Sigma$	(by 2. and **G**)
4. $\Diamond(\Box\varphi \wedge \Diamond\psi) \in \Sigma$	(by 1. and 3., and $Th(K)$)
5. $\Diamond\Diamond(\varphi \wedge \psi) \in \Sigma$	(by $Th(K)$)
6. $\Box\Box\neg(\varphi \wedge \psi) \in \Sigma$	(by (a) and (N))
7. $\neg\Diamond\Diamond(\varphi \wedge \psi) \in \Sigma$	(by Dual).

But since Σ is (maximal) consistent, we get a contradiction. Therefore, (3) is consistent, so R^C is incestual, i.e. the canonical model for **G** is a convergent model; thus **G** is complete.

4. Conclusion

We provided simple proofs for the completeness of the normal modal logics **B**, **K**5 and **G**. We think that we simplified existing proofs, of completeness for these logics, generally complicated.

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Bəzi normal modal məntiqlərin tamlığı haqqda

Gülsah Önər

XÜLASƏ

İşdə bəzi B, K5 və G normal modal məntiqlərin tamlığının sadə isbatları verilir.

Açar sözlər: Tamlıq, normal modal məntiq.

О полноте некоторых нормальных модальных логик

Гюлшах Онер

РЕЗЮМЕ

В работе дается простые доказательства полноты некоторых нормальных модальных логик B, K5 и G.

Ключевые слова: Полнота, нормальная модальная логика